

An Identity Involving Integration with Respect to Variable Order of Fractional Derivative

Ivan Matychyn

1 Introduction

In 1993, Samko and Ross [1] introduced the study of fractional integration and differentiation when the order is not a constant but a function. This suggestion gave rise to a number of further ideas and results [2, 3, 4]. In particular, this implies a possibility of integration with respect to derivative's order. Here an identity is presented, in which an expression involving Riemann–Liouville fractional derivative is integrated with respect to the derivative's order.

2 Preliminaries

Definition 1. Suppose $f : [a, \infty) \rightarrow \mathbb{R}$ is an absolutely continuous function. The Riemann–Liouville (left-sided) fractional integral and derivative of order α , $m - 1 < \alpha < m$, $m \in \mathbb{N}$, are defined as follows:

$$J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > a,$$

$$D_{a+}^{\alpha} f(t) = \frac{d^m}{dt^m} J_{a+}^{m-\alpha} f(t), \quad t > a.$$

In what follows we will omit the lower limit of integration in the notation if it is equal to zero, i.e. $J^{\alpha} f(t) \triangleq J_{0+}^{\alpha} f(t)$, $D^{\alpha} f(t) \triangleq D_{0+}^{\alpha} f(t)$.

Here $\Gamma(\alpha)$ denotes Euler's Gamma function.

Property 1 (Property 2.1 [5]). Let $\alpha, \beta > 0$. Then the following identity holds:

$$D_a^{\alpha} (t - a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (t - a)^{\beta-\alpha-1}.$$

In particular, it follows from Proposition 1, that

$$D^{\alpha} 1 = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}, \tag{1}$$

$$D^{\alpha} t^{n-1} = \frac{\Gamma(n)}{\Gamma(n - \alpha)} t^{n-\alpha-1}, \quad n \in \mathbb{N}. \tag{2}$$

Definition 2. The sinc function (“Cardinal Sine”) is defined as follows:

$$\text{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Property 2.

$$\int_{-\infty}^{\infty} \text{sinc}(x) dx = 1. \quad (3)$$

The following properties of Euler’s Gamma function will be used in the sequel.

Property 3.

$$\Gamma(1 + \alpha)\Gamma(1 - \alpha) = \alpha\Gamma(\alpha)\Gamma(1 - \alpha) = \frac{\pi\alpha}{\sin(\pi\alpha)} = \frac{1}{\text{sinc}(\alpha)}, \quad (4)$$

Property 4.

$$\Gamma(z + n) = (z)_n \Gamma(z),$$

where $(z)_n = z(z + 1) \dots (z + n - 1)$ is the Pochhammer symbol.

The following theorem allowing to calculate some improper integrals with the help of contour integrals in the complex plane will also be used in the sequel.

Theorem 1 (Indented Trigonometric Integrals [6]). Assume that $P(z)$, $Q(z)$, $z \in \mathbb{C}$, are polynomials with real coefficients of degree m and n , respectively, where $n \geq m + 1$ and that $Q(z)$ has simple zeros at the points t_1, \dots, t_L on the x -axis. If p is a positive real number, and if $f(z) = \frac{e^{ipz} P(z)}{Q(z)}$, then we can compute the Cauchy Principal Value (P.V.) of the following integral

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(px) dx = \text{Im} \left(2\pi i \sum_{j=1}^k \text{res}_{z=z_j} f(z) + \pi i \sum_{j=1}^L \text{res}_{z=t_j} f(z) \right).$$

3 The Main Result

Lemma 1. For every $n = 1, 2, \dots$ the following identity holds true:

$$\int_{-\infty}^{\infty} \frac{t^\alpha}{\Gamma(\alpha + 1)} (D^\alpha t^{n-1}) d\alpha = (2t)^{n-1}. \quad (5)$$

Proof. For $n = 1$, in view of Properties 3 and 2, we have

$$\int_{-\infty}^{\infty} \frac{t^\alpha}{\Gamma(1 + \alpha)} (D^\alpha 1) d\alpha = \int_{-\infty}^{\infty} \frac{\sin(\pi\alpha)}{\pi\alpha} d\alpha = \int_{-\infty}^{\infty} \text{sinc}(\alpha) d\alpha = 1.$$

Now suppose $n = 2, 3, \dots$. Then, by virtue of Properties 3 and 4, we have

$$\Gamma(n - \alpha)\Gamma(1 + \alpha) = (1 - \alpha)_{n-1} \Gamma(1 - \alpha)\Gamma(1 + \alpha) = (1 - \alpha)_{n-1} \frac{\pi\alpha}{\sin(\pi\alpha)},$$

hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{t^\alpha}{\Gamma(1+\alpha)} (D^\alpha t^{n-1}) d\alpha &= t^{n-1} \int_{-\infty}^{\infty} \frac{\Gamma(n)}{\Gamma(n-\alpha)\Gamma(1+\alpha)} d\alpha \\
&= t^{n-1} \int_{-\infty}^{\infty} \binom{n-1}{\alpha} d\alpha \\
&= \frac{t^{n-1}}{\pi} (n-1)! \int_{-\infty}^{\infty} \frac{\sin(\pi\alpha)}{\alpha(1-\alpha)_{n-1}} d\alpha.
\end{aligned} \tag{6}$$

It follows from Theorem 1 that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(\pi\alpha)}{\alpha(1-\alpha)_n} d\alpha &= \int_{-\infty}^{\infty} \frac{\sin(\pi\alpha)}{\alpha(1-\alpha)(2-\alpha)\dots(n-\alpha)} d\alpha \\
&= \operatorname{Im} \left(\pi i \sum_{j=0}^n \operatorname{res}_{z=j} \frac{e^{i\pi z}}{z(1-z)(2-z)\dots(n-z)} \right).
\end{aligned} \tag{7}$$

Since

$$\begin{aligned}
\operatorname{res}_{z=0} \frac{e^{i\pi z}}{z(1-z)(2-z)\dots(n-z)} &= \frac{1}{n!}, \\
\operatorname{res}_{z=j} \frac{e^{i\pi z}}{z(1-z)(2-z)\dots(n-z)} &= \frac{1}{n!} \binom{n}{j}, \quad j = 1, \dots, n.
\end{aligned}$$

we have

$$\int_{-\infty}^{\infty} \frac{\sin(\pi\alpha)}{\alpha(1-\alpha)_n} d\alpha = \frac{\pi}{n!} \sum_{j=0}^n \binom{n}{j} = \frac{\pi 2^n}{n!}. \tag{8}$$

Thus

$$\int_{-\infty}^{\infty} \frac{t^\alpha}{\Gamma(1+\alpha)} (D^\alpha t^{n-1}) d\alpha = \frac{t^{n-1}}{\pi} (n-1)! \frac{\pi 2^{n-1}}{n-1!} = (2t)^{n-1}.$$

□

Corollary 1.

$$\int_{-\infty}^{\infty} \binom{n}{\alpha} d\alpha = 2^n, \quad n \in \mathbb{N}.$$

This Corollary follows from Lemma 1 and (6).

Corollary 2 (Main Identity). *For any function $f(t)$ that is analytic in some neighborhood of zero $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$, the following identity holds true*

$$\int_{-\infty}^{\infty} \frac{t^\alpha}{\Gamma(\alpha+1)} [D^\alpha f(t/2)] d\alpha = f(t), \quad t \in (-\varepsilon, \varepsilon). \tag{9}$$

References

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